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The Trace to Closed Sets of Functions in R^n with Second Difference of Order $O(h)$

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0. INTRODUCTION

0.1. In this paper we generalize the classical Whitney extension theorem to classes of functions defined in terms of second-order differences. The Whitney extension theorem states, in a version given in [8, Chap. VI] that every function in $\mathrm{Lip}(\alpha, F)$, $\alpha > 0$, may be extended to a function in $\mathrm{Lip}(\alpha, R^n)$. Here F is an arbitrary closed set, and $\mathrm{Lip}(\alpha, F)$ is, for $0 < \alpha \leqslant 1$ (see [8, p. 176] for $\alpha > 1$), the space of all functions f satisfying $|f(x) - f(y)| \leqslant M|x - y|^\alpha$, $x, y \in F$, and $|f(x)| \leqslant M$, $x \in F$, for some constant M which may depend on f.

It is well known that in many problems in analysis the space $\mathrm{Lip}(1, R^n)$ can be replaced in a natural way, following [12], by a somewhat larger space which is defined by means of second differences. This larger space $\Lambda_1(R^n) = \Lambda(R^n)$ consists of all continuous functions f satisfying $|f(x - h) - 2f(x) + f(x + h)| \leqslant M|h|$, $x, h \in R^n$, and $|f(x)| \leqslant M$, $x \in R^n$ (see Definition 2.1). We prove, for the space $\Lambda(R^n)$, an analog to the Whitney extension theorem. We define a space $\Lambda_1(F)$ of functions on an arbitrary closed set F (Definition 1.1) and prove that every $f \in \Lambda_1(F)$ may be extended to a function defined in R^n belonging to $\Lambda(R^n)$ (Theorem 3.1). This is the analog for $\Lambda(R^n)$ of Whitney's theorem and the main result of this paper. The converse of this result also holds, i.e., the restriction to F of a function in $\Lambda(R^n)$ belongs to $\Lambda_1(F)$. This is an immediate consequence of the definition of $\Lambda_1(F)$ and Proposition 2.1 below, stating that $\Lambda_1(F)$ coincides with $\Lambda(R^n)$ for $F = R^n$. Thus $\Lambda_1(F)$ is the "trace" of $\Lambda(R^n)$ to F.

A major problem is defining $\Lambda_1(F)$, since one cannot automatically use second differences on a nonconvex set F; $x - h$ and $x + h$ may belong to F but not x (compare also Remark 4.2). The definition of $\Lambda_1(F)$ is given in Section 1, and in Remark 1.3 some hopefully clarifying comments to it are given. The proof of the extension Theorem 3.1 is closely related to the proof of the Whitney extension theorem. For a comparison between the two theorems we refer to Section 3.1.

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The extension of Theorem 3.1 to classes $\Lambda_k(F)$, $k > 1$, k integer (Definition 5.1), of functions defined in terms of higher-order differences instead of second-order differences is treated in Section 5 (Theorem 5.1).

The definition of $\Lambda_1(F)$ is rather implicit, but for some sets it is possible to give simpler (but equivalent) definitions. In Section 2 we give some equivalent definitions of $\Lambda_1(F)$ when $F = R^n$ and in Section 4 we discuss how these can be transferred to more general sets. In particular, we show that if F is the closure of a Lipschitz domain in R^n, then $\Lambda_1(F)$ can again be defined by means of second differences. In this case, Theorem 3.1 may be considered as a special case of earlier results concerning the trace of general Lipschitz or Besov spaces $\Lambda_\alpha^{p,q}(R^n)$ to domains in R^n (Corollary 4.2); see [8, p. 150] for the definition of $\Lambda_\alpha^{p,q}(R^n)$.

Thus Theorem 3.1 is also related to the theory of the trace of general Lipschitz spaces $\Lambda_\alpha^{p,q}(R^n)$, $\alpha > 0$, $1 \leqslant p$, $q \leqslant \infty$, and Sobolev spaces, to domains or linear subvarieties of R^n (see, e.g., [8, Chap. VI] for definitions and such trace problems). Actually, our spaces $\Lambda_k(F)$ coincide, when $F = R^n$, with the Lipschitz spaces $\Lambda_k^{\infty,\infty}(R^n)$ (see Proposition 2.1 for $k = 1$ and Section 5 for $k > 1$); however, in this paper we take as elements in $\Lambda_k(F)$ the continuous representatives of the elements in $\Lambda_k^{\infty,\infty}(R^n)$ as defined in [6]. When α is not an integer, $\Lambda_\alpha^{\infty,\infty}(R^n)$ coincides with Lip(α, R^n) and, consequently, Whitney's extension theorem solves the problem of determining the trace to F of $\Lambda_\alpha^{\infty,\infty}(R^n)$ in this case. This is the reason why we consider the integer case only. Our interest in the problem studied in this paper comes from our work in [4–6]. In these papers we introduced spaces $B_\beta^{p,q}(F)$ for *noninteger* β, $1 \leqslant p, q < \infty$, where F is a rather general closed set, and proved that the spaces $B_\beta^{p,q}(F)$ occur as the trace to F of the classical Lipschitz spaces $\Lambda_\alpha^{p,q}(R^n)$ if $\beta = \alpha - (n - d)/p$ and d is the Hausdorff dimension of F. In a forthcoming publication we shall show how spaces $B_\beta^{p,q}(F)$ may be defined for integers β along the lines of the present paper so that the missing part (the case when β is an integer) of [4–6] is filled in.

Parts of the results of this paper have been presented in [9] in somewhat weaker versions.

0.2. *Notation.* R^n is the n-dimensional Euclidean space $x = (x^1, x^2, \ldots, x^n)$, F is a closed set with boundary ∂F. $d(x, F)$ is the distance from x to F, and $d(E, F)$ is the distance from the set E to F. $\Delta_h{}^2 f(x)$ is the symmetrical second difference of f with step h at x, i.e.,

$$\Delta_h{}^2 f(x) = f(x + h) - 2f(x) + f(x - h).$$

j is always a multiindex, $j = (j_1, j_2, \ldots, j_n)$, and we use the notation

$$|j| = j_1 + j_2 + \cdots + j_n \qquad \text{and} \qquad x^j = (x^1)^{j_1} (x^2)^{j_2} \cdots (x^n)^{j_n}.$$

$D^j f$ and $f^{(j)}$ both denote the partial derivative of f corresponding to j. Df is the gradient of f, and $h\, Df$ is the scalar product $\sum_{|j|=1} h^j D^j f$. c and M denote different constants most of the time they appear.

1. The Space $\Lambda_1(F)$

Spaces $\Lambda_k(F)$, $k > 1$, are defined in Section 5. In the definition below, j denotes an n-dimensional multiindex of length $|j|$. The functions $f_{j\nu}$, $|j| = 1$, in the definition are conveniently thought of as partial derivatives of the function $f_\nu = f_{0\nu}$. Actually, Whitney's definition of "derivatives" on an arbitrary closed set is based upon conditions similar to condition (ii) in the definition below.

Definition 1.1. Let $a > 0$ and let F be a closed subset of R^n. Then f belongs to *the space* $\Lambda_1(F)$ if there exist collections $\{f_{j\nu}\}_{|j|\leqslant 1}$, $\nu = 1, 2,\ldots$, and a constant M, such that for $x, y \in F$ (we put $f_{0\nu} = f_\nu$ whenever it is convenient)

(i) $$|f(x) - f_\nu(x)| \leqslant M2^{-\nu}, \tag{1.1}$$

$$|f_{j\nu}(x) - f_{j\mu}(x)| \leqslant M2^{\mu-\nu}, \qquad \mu \geqslant \nu, |j| = 1, \tag{1.2}$$

(ii) $$\Big|f_\nu(x) - f_\nu(y) - \sum_{|j|=1} (x-y)^j f_{j\nu}(y)\Big| \leqslant M2^{-\nu}, \quad |x-y| \leqslant a2^{-\nu}, \tag{1.3}$$

$$|f_{j\nu}(x) - f_{j\nu}(y)| \leqslant M, \ |x-y| \leqslant a2^{-\nu}, \qquad |j| = 1, \tag{1.4}$$

(iii) $$|f_1(x)| \leqslant M, \tag{1.5}$$

$$|f_{j1}(x)| \leqslant M, \qquad |j| = 1. \tag{1.6}$$

As the *norm* $\|f\|_{\Lambda_1(F)}$ of f, we take infimum of all constants M, such that conditions (i)–(iii) are satisfied for some $\{f_{j\nu}\}_{|j|\leqslant 1}$.

The following remarks are of importance in connection with this definition.

Remark 1.1. From (i) and (iii) it follows that

$$|f_\nu(x)| \leqslant |f_\nu(x) - f(x)| + |f(x) - f_1(x)| + |f_1(x)| \leqslant 2M$$

and that for $|j| = 1$

$$|f_{j\nu}(x)| \leqslant |f_{j\nu}(x) - f_{j1}(x)| + |f_{j1}(x)|$$

$$= \Big|\sum_{i=2}^{\nu} f_{ji}(x) - f_{j(i-1)}(x)\Big| + |f_{j1}(x)| \leqslant (\nu - 1)\, 2M + M \leqslant 2M\nu$$

so for $x \in F$, $\nu = 1, 2, \ldots$, there holds

$$|f_\nu(x)| \leqslant 2M \tag{1.7}$$

and

$$|f_{j\nu}(x)| \leqslant 2M\nu, \qquad |j| = 1. \tag{1.8}$$

In a similar way we can also see that it is enough to assume (1.2) for $\mu = \nu + 1$. We can also easily see that f is continuous on F if $f \in \Lambda_1(F)$. This follows from

$$|f(x) - f(y)| \leqslant |f(x) - f_\nu(x)| + \Big| f_\nu(x) - f_\nu(y) - \sum_{|j|=1} (x-y)^j f_{j\nu}(y) \Big|$$
$$+ \Big| \sum_{|j|=1} (x-y)^j f_{j\nu}(y) \Big| + |f_\nu(y) - f(y)| \leqslant c2^{-\nu} + c\nu 2^{-\nu}$$

if $x, y \in F$ and $|x - y| \leqslant a2^{-\nu}$. Here the last inequality is a consequence of (1.1), (1.3), and (1.8).

Remark 1.2. Different values on the constant a appearing in Definition 1.1 give raise to equivalent norms. To see this, let $a_1 < a_2$, and denote the corresponding norms by $\| \cdot \|_{a_1}$ and $\| \cdot \|_{a_2}$. Then clearly $\| \cdot \|_{a_1} \leqslant \| \cdot \|_{a_2}$.

In order to deduce a converse inequality, let N be an integer such that $2^N a_1 > a_2$, and consider the collections $\{g_{j\nu}\}$ given by $g_{j\nu} = f_{j,\nu-N}$, $\nu > N$, and by $g_{j\nu} = f_{j1}$, $\nu = 1, 2, \ldots, N$, where $\{f_{j\nu}\}$ is a collection satisfying (i)–(iii) in Definition 1.1 with $a = a_1$ and $M = 2\|f\|_{a_1}$. Then $\{g_{j\nu}\}$ satisfies (i)–(iii) with $a = 2^N a_1$, $M = C\|f\|_{a_1}$ (and $f_{j\nu}$ replaced by $g_{j\nu}$), where C is a constant depending only on n, a_1, and N. Thus we have $\| \cdot \|_{a_2} \leqslant \| \cdot \|_{2^N a_1} \leqslant C \| \cdot \|_{a_2}$.

Remark 1.3. It is possible to give several equivalent definitions of $\Lambda_1(F)$ (compare also Remark 4.5). For example, *if* $1 < \alpha \leqslant 2$ *and we replace* (ii) *in Definition* 1.1 *by*

$$\Big| f_\nu(x) - f_\nu(y) - \sum_{|j|=1} (x-y)^j f_{j\nu}(y) \Big| \leqslant M 2^{\nu(\alpha-1)} |x-y|^\alpha, \qquad x, y \in F \tag{1.9}$$

and

$$|f_{j\nu}(x) - f_{j\nu}(y)| \leqslant M 2^{\nu(\alpha-1)} |x-y|^{\alpha-1}, \quad x, y \in F \tag{1.10}$$

we obtain an equivalent definition. (It is obvious that (ii) in Definition 1.1 follows from these inequalities. Conversely, if $f \in \Lambda_1(F)$ as defined in Definition 1.1, then we may extend f by means of Theorem 3.1 and obtain a function $Ef \in \Lambda(R^n)$.

The methods used in proving Proposition 2.1 below now give us $\{f_{j\nu}\}_{|j|\leqslant 1}$ with the desired properties, since (1.9) now follows from (1.10), and (1.10) follows from (2.5) if $|x-y| \leqslant 2^{-\nu}$, from (1.2) and (1.4) if $2^{-\nu} < |x-y| \leqslant 1$, and from (1.8) if $|x-y| > 1$. In the middle case, take μ so that $2^{-\mu-1} < |x-y| \leqslant 2^{-\mu}$. Then

$$\begin{aligned}
&|f_{j\nu}(x) - f_{j\nu}(y)| \\
&\quad \leqslant |f_{j\nu}(x) - f_{j\mu}(x)| + |f_{j\nu}(y) - f_{j\mu}(y)| + |f_{j\mu}(x) - f_{j\mu}(y)| \\
&\quad \leqslant 4M(\nu - \mu) + M \leqslant cM2^{(\nu-\mu)(\alpha-1)}
\end{aligned}$$

(cf. Remark 1.1 for the second to last estimate). Conditions (1.9) and (1.10) together with the boundedness (1.7) and (1.8) of $f_{j\nu}$, $|j| \leqslant 1$, mean that $\{f_{j\nu}\}_{|j|\leqslant 1} \in \mathrm{Lip}(\alpha, F)$ with norm in $\mathrm{Lip}(\alpha, F)$ less than $M2^{\nu(\alpha-1)}$. This means that, *for* $1 < \alpha \leqslant 2$, *conditions* (ii) *and* (iii) *in the definition of* $\Lambda_1(F)$ *may be replaced by the assumption that* $\{f_{j\nu}\}_{|j|\leqslant 1} \in \mathrm{Lip}(\alpha, F)$ *with norm in* $\mathrm{Lip}(\alpha, F)$ *less than* $M2^{\nu(\alpha-1)}$. Consequently, we get an alternative definition of $\Lambda_1(F)$ based upon approximation with smoother functions in the class $\mathrm{Lip}(\alpha, F)$, which is in a natural way defined on an arbitrary closed set.

It is useful to have the weaker assumptions (1.3) and (1.4) instead of (1.9) and (1.10), for example, in the proof of Proposition 4.2.

2. Different Definitions of $\Lambda_1(R^n)$

2.1. We shall start by showing that $\Lambda_1(F)$ for $F = R^n$ coincides with the class $\Lambda(R^n)$ of functions satisfying the following smoothness condition.

Definition 2.1. The function f belongs to the class $\Lambda(R^n)$ if f is continuous on R^n and for some constant M, $|f(x)| \leqslant M$ and $|\Delta_h^2 f(x)| \leqslant M|h|$ for $x, h \in R^n$.

The *norm* of $f \in \Lambda(R^n)$ is the infimum of the constants M.

Proposition 2.1. $\Lambda_1(R^n) = \Lambda(R^n)$ *with equivalent norms.*

Proof. (1) Suppose that $f \in \Lambda(R^n)$. Take a function $\varphi \in C_0^\infty(R^n)$ such that $\varphi(x) = 0$ if $|x| \geqslant 1$, $\int \varphi\, dx = 1$, and $\varphi \geqslant 0$. We also assume that $\varphi(x) = \varphi(-x)$ which gives that $(D^j\varphi)(x) = (D^j\varphi)(-x)$ if $|j| = 2$. Put

$$\varphi_r(x) = r^{-n}\varphi(x/r), \qquad r > 0,$$

and define f_r by

$$f_r(x) = (f * \varphi_r)(x) = \int f(x-t)\,\varphi_r(t)\,dt = \int f(x+t)\,\varphi_r(t)\,dt.$$

Then, since $\int \varphi_r(x)\,dx = 1$ and $\varphi_r(x) = 0$ if $|x| \geqslant r$,

$$2(f_r(x) - f(x)) = \int_{|t|\leqslant r} (f(x+t) + f(x-t) - 2f(x))\,\varphi_r(t)\,dt$$

which by the assumption is less than Mr. Hence

$$|f_r(x) - f(x)| \leqslant cr, \qquad x \in R^n, \quad r > 0. \tag{2.1}$$

Next we shall prove that

$$|D^j f_r(x)| \leqslant cr^{-1}, \qquad |j| = 2, \quad x \in R^n, \quad r > 0. \tag{2.2}$$

In fact, since $(D^j\varphi_r)(x) = (D^j\varphi_r)(-x)$ and $\int D^j\varphi_r(x)\,dx = 0$ for $|j| = 2$, we get

$$2D^j f_r(x) = \int_{|t|\leqslant r} (f(x-t) + f(x+t) - 2f(x))\,D^j\varphi_r(t)\,dt.$$

But $D^j\varphi_r(t) = r^{-n-2}(D^j\varphi)(t/r)$, $|j| = 2$, and so

$$|2D^j f_r(x)| \leqslant \int_{|t|\leqslant r} M\,|t|\,r^{-n-2}c\,dt = cr^{-1},$$

which is (2.2). We remark in passing that we also get, for $x \in R^n$ and $|j| = 1$,

$$|f_r(x)| \leqslant M \qquad \text{and} \qquad |D^j f_r(x)| = \left| \int f(t)\,D^j\varphi_r(x-t)\,dt \right| \leqslant cr^{-1}. \tag{2.3}$$

From (2.2) and the mean-value theorem we now obtain for $|h| \leqslant r$, $(x \in R^n, r > 0)$

$$|f_r(x+h) - f_r(x) - hDf_r(x)| \leqslant c\,|h|^2\,r^{-1} \leqslant cr \tag{2.4}$$

and

$$|Df_r(x) - Df_r(x+h)| \leqslant c\,|h|\,r^{-1} \leqslant c. \tag{2.5}$$

We finally want to prove that

$$|D^j f_{r_1}(x) - D^j f_{r_2}(x)| \leqslant cr_2/r_1, \qquad r_2 > r_1, x \in R^n, |j| = 1. \tag{2.6}$$

In fact, by inserting suitable terms we find

$$\begin{aligned}
&|h(Df_{r_1}(x) - Df_{r_2}(x)| \\
&\quad \leqslant |f_{r_2}(x+h) - f_{r_2}(x) - hDf_{r_2}(x)| \\
&\qquad + |-f_{r_1}(x+h) + f_{r_1}(x) + hDf_{r_1}(x)| + |f_{r_1}(x+h) - f_{r_2}(x+h)| \\
&\qquad + |f_{r_2}(x) - f_{r_1}(x)|,
\end{aligned}$$

and if we estimate the first two terms in the right-hand side by means of (2.4) and the last two by means of (2.1) (by inserting f), we get the estimate $c(r_1 + r_2)$ if $|h| \leqslant r_1$. By taking $h = r_1 e_i$, where e_i is the unit vector in the x^i-direction, we obtain (2.6).

In order to see that $f \in \Lambda_1(R^n)$ it is now enough to define f_ν by f_r with $r = 2^{-\nu}$. The estimates (2.1)–(2.6) show that $f \in \Lambda_1(R^n)$ and that

$$\|f\|_{\Lambda_1(R^n)} \leqslant c\,\|f\|_{\Lambda(R^n)}.$$

(2) Conversely, let $f \in \Lambda_1(R^n)$ and let $\{f_{j\nu}\}_{|j|\leqslant 1}$, $\nu = 1, 2,\ldots$, be given by Definition 1.1 (with $F = R^n$). Then f is continuous and bounded by (1.1) and (1.5) in Definition 1.1. Furthermore,

$$\Delta_h{}^2 f(x) = (\Delta_h{}^2 f(x) - \Delta_h{}^2 f_\nu(x)) + \Delta_h{}^2 f_\nu(x) = \mathrm{I} + \mathrm{II}. \tag{2.7}$$

Choose ν such that $2^{-\nu} < |h| \leqslant 2^{-\nu+1}$ (if $0 < |h| \leqslant 1$; otherwise it is trivial that $|\Delta_h{}^2 f(x)| \leqslant c\,|h|$).

From (1.1) we conclude that $|I| \leqslant c2^{-\nu} \leqslant c\,|h|$, and from (1.3) that

$$|\,\mathrm{II}\,| \leqslant \Big|f_\nu(x+h) - f_\nu(x) - \sum_{|j|=1} h^j f_{j\nu}(x)\Big| + \Big|f_\nu(x-h) - f_\nu(x) + \sum_{|j|=1} h^j f_{j\nu}(x)\Big| \leqslant c2^{-\nu} \leqslant c\,|h|\,.$$

Hence, $f \in \Lambda(R^n)$ and $\|f\|_{\Lambda(R^n)} \leqslant c\,\|f\|_{\Lambda_1(R^n)}$ and the proposition is proved.

Remark 2.1. In the second part of the proof we used only (1.1), (1.3), and (1.5) to infer that $f \in \Lambda(R^n)$. This means that (1.1), (1.3), and (1.5) in Definition 1.1 imply (1.2), (1.4), and (1.6) when $F = R^n$. This is not true for a general F which is seen from the following example.

EXAMPLE. Let $0 < \beta < 1$, and put

$$F = \{0\} \cup \{a_n = 2 \cdot 2^{-n}, n \geqslant 1\} \cup \{b_n = 2 \cdot 2^{-n} + 2^{-n/\beta}, n \geqslant 1\}.$$

Define f on F by $f(x) = 2^{-n}$, $x = b_n$ and $f(x) = 0$ elsewhere. Then no $Ef \in \Lambda(R)$ can coincide with f on F, since a function Ef in $\Lambda(R)$ satisfies $|Ef(x) - Ef(y)| \leqslant c\,|x-y|\,|\ln|x-y||$ (see [11, p. 44]), but this is not satisfied by f on F, since $|f(b_n) - f(a_n)| = |b_n - a_n|^\beta$. On the other hand, f satisfies all conditions in Definition 1.1 except (1.2), if we define $f_{j\nu}$ for $j = 0$ and $j = 1$ by $f_{0\nu}(x) = 0$, $x \leqslant b_\nu$, $f_{0\nu}(x) = f(x)$, $x > b_\nu$, $f_{1\nu}(x) = 0$, $x \leqslant b_\nu$, and $f_{1\nu}(x) = 2^{-n}/2^{-n/\beta}$, $x = a_n$ and $x = b_n$, $n < \nu$.

It is obvious that (1.1), (1.5), and (1.6) are satisfied, and that the inequalities in (1.3) and (1.4) are satisfied if $x, y \leqslant b_\nu$ or $x = a_n$, $y = b_n$ for some n, which is always the situation if $|x-y| \leqslant 2^{-\nu}$, $x, y \in F$.

2.2. There are several other equivalent ways to define $\Lambda(R^n)$. We shall state two of them. The first, given by Proposition 2.2, is of interest here, since it is in spirit very similar to, but simpler than, our definition of $\Lambda_1(R^n)$. However, it cannot be used to define $\Lambda_1(F)$. It is more or less well known, cf. Remark 4.3 below.

PROPOSITION 2.2. *$f \in \Lambda(R^n)$ if and only if for every $r > 0$ there exists a function $f_r \in C^2(R^n)$ such that for $x \in R^n$ and $r > 0$,*

$$|f_r(x)| \leqslant M, \tag{2.8}$$

$$|D^j f_r(x)| \leqslant Mr^{-1}, \qquad |j| = 2, \tag{2.9}$$

and

$$|f_r(x) - f(x)| \leqslant Mr. \tag{2.10}$$

Furthermore, the norm of f in $\Lambda(R^n)$ is equivalent to the infimum of the constants M.

Proof. The "only if" part follows from the proof of Proposition 2.1 (with $f_r = f * \varphi_r$). The "if" part follows almost exactly as in the proof of Proposition 2.1 by using (2.7) with f_ν changed to f_r, where $r = |h|$, and then estimating II in (2.7) by means of the mean-value theorem and (2.9).

2.3. The characterization of $\Lambda(R^n)$ given by the next proposition, will be generalized to more general sets in Section 4. It is a consequence of known results concerning polynomial approximation.

PROPOSITION 2.3. *$f \in \Lambda(R^n)$ if and only if f is continuous and, for some constant M,*

$$|f(x)| \leqslant M, \qquad x \in R^n, \tag{2.11}$$

and, if $x_0, x_1, \dots, x_n$ are $n + 1$ affine independent points (in the sense that the vectors $x_1 - x_0, x_2 - x_0, \dots, x_n - x_0$ are linearly independent) and P is the polynomial of the first degree in n variables interpolating to f at $x_0, x_1, \dots, x_n$, then

$$|f(x) - P(x)| \leqslant M \max_{0 \leqslant i,k \leqslant n} |x_i - x_k|, \tag{2.12}$$

for all points x belonging to the convex hull K of $\{x_0, \dots, x_n\}$. The norm of f in $\Lambda(R^n)$ is equivalent to the infimum of the constants M.

Proof. Let ρ be the diameter of K, let Ω be a sphere of diameter $c_1\rho$ containing K, and let $f \in \Lambda(R^n)$. From [1] (see also [3]), it follows that there exists a polynomial P of degree 1 such that $\| f - P \|_{\infty,\Omega} \leqslant \omega \sup_h \| \Delta_h^2 f \|_{\infty,\Omega}$, where the norm on the right is taken over $x \in \Omega$ such that $x - h$ and $x + h$ both are in Ω. Here ω is a constant, depending only on n. Thus,

$$\| f - P \|_{\infty,\Omega} \leqslant cN\rho, \tag{2.13}$$

where c depends on c_1 and n, and N is the $\Lambda(R^n)$-norm of f. Consider now the linear functional F_x given by $F_x(g) = P_1 g(x)$, $g \in C(\Omega)$, where $P_1 g$ is the polynomial of degree 1 interpolating to g at $x_0, x_1, \ldots, x_n$, and let $I_x(g) = g(x)$. Then

$$\begin{aligned} | f(x) - P_1 f(x)| &= |(I_x - F_x) f| = |(I_x - F_x)(f - P)| \\ &\leqslant (1 + \| F_x \|)(\| f - P \|_{\infty,\Omega}). \end{aligned}$$

If $x \in K$, then $\| F_x \| \leqslant 1$, so combined with (2.13) this gives (2.12).

In order to prove the converse we just note that if P is a first-degree polynomial interpolating to f at $n + 1$ suitable points, two of which are $x + h$ and $x - h$, then

$$| \Delta_h^2 f(x)| = | \Delta_h^2 f(x) - \Delta_h^2 P(x)| = 2 \, | f(x) - P(x)| \leqslant c \, | h |.$$

This completes the proof of the proposition.

Remark. Instead of appealing to the results in, e.g., [1] in the proof above, a polynomial P satisfying (2.13) may be obtained by taking P as the first-degree Taylor polynomial of f_ρ at, e.g., x_0, where f_ρ is as in Proposition 2.2.

In the proof of Proposition 2.3, we actually obtain (2.12) for all x in Ω, but then M depends on $\| F_x \|$, which depends on the shape of K; it is easy to see that $\| F_x \| \leqslant c\rho/\rho_1$, where ρ_1 is the diameter of the sphere inscribed in K, and c depends on c_1 and n. In particular, for $n = 2$ we obtain the following result, which will be referred to later on.

Let v be fixed, $0 < v < \pi/3$, and let x_0, x_1, and x_2 be v-uniformly affine independent in the following sense:

The angles in the triangle Δ with corners x_0, x_1 and x_2, are all larger than or equal to v. (2.14)

If P is the first degree polynomial interpolating to $f \in \Lambda(R^n)$ at x_0, x_1, x_2, then

$$| f(x) - P(x)| \leqslant M \max_{0 \leqslant i,k \leqslant 2} | x_i - x_k | \tag{2.15}$$

with M depending on c, for x such that

$$\max_{0 \leqslant i \leqslant 2} | x - x_i | \leqslant c \max_{0 \leqslant i,k \leqslant 2} | x_i - x_k | .$$

It is easy to see that (2.15) does not hold if we omit condition (2.14).

3. The Extension Theorem

3.1. We shall prove the following extension theorem.

Theorem 3.1. *Let F be a closed subset of R^n. Then every $f \in \Lambda_1(F)$ may be extended to a function Ef in $\Lambda(R^n)$. Furthermore, the extension can be made so that $\| Ef \|_{\Lambda(R^n)} \leqslant c \| f \|_{\Lambda_1(F)}$, where the constant c only depends on n, and so that Ef is infinitely differentiable outside F.*

Conversely, it follows from Proposition 2.1 that the restriction to F of a function in $\Lambda(R^n)$ belongs to $\Lambda_1(F)$, and the norm in $\Lambda_1(F)$ is less than a constant times the norm in $\Lambda(R^n)$. Together with Theorem 3.1 this gives:

Main Result. *The trace to F of $\Lambda(R^n)$ is $\Lambda_1(F)$.*

Before going into detail, we make a brief sketch of the proof. Let $f \in \Lambda_1(F)$ be given, and let $\{f_{j\nu}\}_{|j| \leqslant 1}$, $\nu = 1, 2, 3, \ldots$, be associated to f as in Definition 1.1. To each $\{f_{j\nu}\}_{|j| \leqslant 1}$, we shall associate a function $\tilde{f}_\nu$ defined on R^n (see Section 3.3). These functions $\tilde{f}_\nu$ are then put together by means of a partition of unity on the layers

$$\Delta_\nu = \{x \mid 2^{-(\nu+1)} < d(x, F) \leqslant 2^{-\nu}\} \tag{3.1}$$

in the following way. Let φ_ν be the nonnegative C^∞-functions equal to zero outside $\Delta_{\nu-1} \cup \Delta_\nu \cup \Delta_{\nu+1}$ with $\sum \varphi_\nu(x) = 1$, $x \in \mathscr{C}F$, given by Lemma 3.1 below ($\mathscr{C}$ denotes complement in R^n). Define Ef by

$$Ef(x) = \sum_{\nu=1}^{\infty} \varphi_\nu(x) \tilde{f}_\nu(x), \; x \in \mathscr{C}F, \qquad \text{and} \qquad Ef(x) = f(x), \; x \in F.$$

Then Ef belongs to $\Lambda(R^n)$; this is shown in Section 3.5, the proof being based upon a number of estimates given in Section 3.4.

This method is, as previously mentioned, closely related to the proof of the Whitney extension theorem. Actually, at least if we used the stronger definition in Remark 1.3, then we could use the extension operator E_1 used in the Whitney extension theorem for Lip(α, F), $\alpha = 2$, (see [8, p. 176]), and define an extension $\bar{E}f$ of $f \in \Lambda_1(F)$ by

$$\bar{E}f(x) = \sum_{1}^{\infty} \varphi_\nu(x) \, E_1(\{f_{j\nu}\}_{|j| \leqslant 1}).$$

Then $\bar{E}f$ is more or less equivalent to the extension Ef sketched above. However, the functions $\tilde{f}_\nu$ are simpler than $E_1(\{f_{j\nu}\}_{|j| \leqslant 1})$, and it seems more natural to use them in our context.

It should be noted that if we use the definition of $\Lambda_1(F)$ given in Remark 1.3 and the extension $\bar{E}f$, then the proof of Theorem 3.1 may be shortened considerably. The estimates corresponding to those in Lemma 3.2 and Lemma 3.3 may then be derived from Stein's version in [8] of the Whitney extension theorem (cf. [9] and the proof of Proposition 4.1 below).

Thus, our main contribution with Theorem 3.1 to the theory of extension of functions, seems to be how to define $\Lambda_1(F)$, letting the definition of Ef on the distance of magnitude $2^{-\nu}$ from F be based upon the approximation $\{f_{j\nu}\}_{|j|\leqslant 1}$ of f, and maybe also the use of the weak assumption (ii) in Definition 1.1 (cf. Remark 1.3).

For simple sets F it is of course possible to use simpler extension operators. For instance, if $f \in \Lambda(R^n)$ and $u(x, y) = (P_y * f)(x)$, $x \in R^n$, $y > 0$, i.e., $(x, y) \in R_+^{n+1}$, denotes the Poisson integral of f, then the second difference $\Delta_h{}^2 u(x, y)$ is $O(h)$ in R_+^{n+1}. Furthermore, straightforward computations show that the extension of f defined by

$$\tilde{f}(x, y) = u(x, y) - y\,\frac{\partial u(x, y)}{\partial y} \qquad \text{for } y > 0$$

and by reflection, $\tilde{f}(x, y) = \tilde{f}(x, -y)$ for $y < 0$, belongs to $\Lambda(R^{n+1})$.

3.2. When we define the extension Ef in Section 3.3, we shall use some partitions of unity, which we describe in this section.

LEMMA 3.1. *Let F be a closed set, and let Δ_ν be given by* (3.1). *Then there exist functions φ_ν, $\nu = \ldots, -2, -1, 0, 1, 2, \ldots$, such that $\varphi_\nu \in C^\infty$, $\varphi_\nu \geqslant 0$, $\varphi_\nu(x) = 0$ if $x \notin \Delta_{\nu-1} \cup \Delta_\nu \cup \Delta_{\nu+1}$, $\sum \varphi_\nu(x) = 1$ if $x \in \mathscr{C}F$, and for all j,*

$$|\varphi_\nu^{(j)}(x)| \leqslant c 2^{\nu|j|} \tag{3.2}$$

where c is a constant only depending on j and n.

Proof. Let φ be a nonnegative function in $C^\infty(R^n)$ supported by $\{x \mid |x| \leqslant 1\}$ with $\int \varphi\, dx = 1$, and define φ_ν by $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$. Then φ_ν is supported by $\{x \mid |x| \leqslant 2^{-\nu}\}$, $\int \varphi_\nu\, dx = 1$, and $|\varphi_\nu^{(j)}(x)| \leqslant 2^{\nu(n+|j|)} M_j$, where $M_j = \max |\varphi^{(j)}|$.

Now, let g_ν be defined by $g_\nu(x) = 1$ if

$$2^{-(\nu+1)} - 2^{-(\nu+3)} < d(x, F) = 2^{-\nu} + 2^{-(\nu+3)}$$

and $g_\nu(x) = 0$ elsewhere, and h_ν by

$$h_\nu(x) = \int g_\nu(t)\, \varphi_{\nu+3}(x - t)\, dt.$$

Then from the above mentioned properties of φ_ν, we easily obtain that $h_\nu = 1$ if $x \in \Delta_\nu$, $h_\nu = 0$ if $x \notin \Delta_{\nu-1} \cup \Delta_\nu \cup \Delta_{\nu+1}$, and that

$$| h_\nu^{(j)}(x)| = \left| \int g_\nu(t)\, \varphi_{\nu+3}^{(j)}(x-t)\, dt \right| \leqslant M_j 2^{(\nu+3)(n+|j|)} 2^{-(\nu+3)n} \omega_n ,$$

where ω_n is the volume of the n-dimensional unit sphere. So we have

$$| h_\nu^{(j)}(x)| \leqslant c 2^{\nu |j|}, \tag{3.3}$$

where c depends on j and n.

Finally, put $\varphi_\nu = h_\nu / \sum_k h_k$. Then φ_ν satisfies the conditions of the lemma; apart from condition (3.2) this is immediate. To realize that (3.2) holds, we put $g = \sum h_k$. Then $(g\varphi_\nu)^{(j)} = h_\nu^{(j)}$, and it follows that $\varphi_\nu^{(j)} g$ is the sum of $h_\nu^{(j)}$ and terms of type $c g^{(j_1)} \varphi_\nu^{(j_2)}$, where $j_1 + j_2 = j$, $j_1 \neq 0$. If we now assume that (3.2) is proved for $|j| < k$, it is easy to obtain, for $|j| = k$, that $|\varphi_\nu^{(j)} g| \leqslant c 2^{\nu|j|}$, from which (3.2) follows, since $g \geqslant 1$, $x \in \mathscr{C}F$. This concludes the proof of Lemma 3.1.

Next we turn to the definition of a family $\{\varphi_{\nu i}\}$ of functions which will be needed in Section 3.3. For fixed ν, the functions $\varphi_{\nu i}$, $i = 1, 2,\ldots$, form a partition of unity based upon certain cubes $Q_{\nu i}$, which are obtained as follows. Divide R^n into closed cubes with sides of length $2^{-\nu}$ parallel to the axes in such a way that the vertices of the cubes have coordinates of the form $m2^{-\nu}$, where m is an integer. Denote these cubes by $Q_{\nu i}$, $i = 1, 2, 3,\ldots$. In order to define the functions $\varphi_{\nu i}$, let Q and $(1+\epsilon)Q$ denote the cubes centered at the origin with sides parallel to the axes of length 1 and $1+\epsilon$, respectively. Let $0 < \epsilon < 2$, and let ψ be a C^∞-function satisfying $0 \leqslant \psi \leqslant 1$, $\psi(x) = 1$ if $x \in Q$, and $\psi(x) = 0$ if $x \notin (1+\epsilon)Q$. Denote the center of $Q_{\nu i}$ by $x_{\nu i}$, and define $\psi_{\nu i}$ by $\psi_{\nu i}(x) = \psi(2^\nu(x - x_{\nu i}))$. Finally, put $\varphi_{\nu i} = \psi_{\nu i}/\sum \psi_{\nu k}$. Then it is easily seen that the functions $\varphi_{\nu i}$ have the following properties: $0 \leqslant \varphi_{\nu i} \leqslant 1$, $\varphi_{\nu i}(x) = 0$ if x belongs to a cube $Q_{\nu m}$ not touching $Q_{\nu i}$, $\sum_i \varphi_{\nu i}(x) = 1$ and (cf. the proof of (3.2))

$$| \varphi_{\nu i}^{(j)}(x)| \leqslant c 2^{|j|\nu}, \qquad \nu, i = 1, 2, 3,\ldots, \tag{3.4}$$

where the constant c depends only on j and n.

3.3. *The extension Ef.* Let $f \in \Lambda_1(F)$ be given, and associate $\{f_{j\nu}\}_{|j| \leqslant 1}$, $\nu = 1, 2, 3,\ldots$, to f as in Definition 1.1 with $M = 2\|f\|_{\Lambda_1(F)}$, and put $(f_\nu = f_{0\nu})$

$$P_\nu(x, y) = f_\nu(y) + \sum_{|j|=1} (x-y)^j f_{j\nu}(y), \qquad x \in R^n, y \in F. \tag{3.5}$$

Let $Q_{\nu i}$ and $\varphi_{\nu i}$ be as defined in the end of Section 3.2, and let $p_{\nu i}$ denote a point in F with $d(p_{\nu i}, Q_{\nu i}) = d(F, Q_{\nu i})$. To $\{f_{j\nu}\}_{|j|\leqslant 1}$ we now associate the function $\tilde{f}_\nu$ given by

$$\tilde{f}_\nu(x) = \sum_i \varphi_{\nu i}(x)\, P_\nu(x, p_{\nu i}), \qquad x \in R^n.$$

Next let $\{\varphi_\nu(x)\}$ be the partition of unity given by Lemma 3.1. We define the extension Ef of f by

$$\begin{aligned} Ef(x) &= \sum_{\nu=1}^{\infty} \varphi_\nu(x)\, \tilde{f}_\nu(x), \qquad && x \in \mathscr{C}F, \\ &= f(x), && x \in F. \end{aligned}$$

It is obvious from the definition that Ef is infinitely differentiable outside F. In the Sections 3.4 and 3.5 we show that $Ef \in \Lambda_1(R^n)$. Actually it will follow from our proof that $\| Ef \|_{\Lambda_1(R^n)} \leqslant c$, where the constant c is independent of f and F as long as $\| f \|_{\Lambda_1(F)} = 1$. This enables us to conclude that in general $\| Ef \|_{\Lambda_1(R^n)} \leqslant c \| f \|_{\Lambda_1(F)}$, where c is independent of f and F.

3.4. In this section we derive estimates on $\tilde{f}_\nu$ and Ef, from which it will easily follow that $Ef \in \Lambda_1(R^n)$. In order to make these estimates easier to survey, we state them in a series of lemmas. However, we first note a couple of facts, which will be used repeatedly below.

Let $P_\nu(x, y)$ be given by (3.5). Then for $x \in R^n$, $q, r \in F$, we have (cf., e.g., [8, p. 177])

$$P_\nu(x, q) - P_\nu(x, r) = f_\nu(q) - P_\nu(q, r) + \sum_{|j|=1} (f_{j\nu}(q) - f_{j\nu}(r))(x - q)^j.$$

Consequently, since we assume that $\{f_{j\nu}\}_{|j|\leqslant 1}$ satisfies (ii) in Definition 1.1, we have

$$| P_\nu(x, q) - P_\nu(x, r)| \leqslant c2^{-\nu} \qquad \text{if} \quad | q - r | \leqslant a2^{-\nu}, \quad | x - q | < a2^{-\nu} \tag{3.6}$$

and also

$$|f_{j\nu}(q) - f_{j\nu}(r)| \leqslant c \qquad \text{if} \quad | q - r | \leqslant a2^{-\nu}, \quad | j | = 1, \tag{3.7}$$

where the constant a may be taken arbitrarily large by Remark 1.2.

We shall also need the following estimate. Let $x \in Q_{\nu m}$, where $Q_{\nu m}$ is a cube touching $Q_{\nu i}$. Then it is easy to realize that

$$| x - p_{\nu i} | \leqslant 2(n^{1/2})\, 2^{-\nu} + d(Q_{\nu i}, F)$$

and that

$$d(Q_{\nu i}, F) \leqslant d(x, Q_{\nu i}) + d(x, F) \leqslant n^{1/2}2^{-\nu} + d(x, F).$$

This gives

$$|x - p_{\nu i}| \leqslant d(x, F) + 3(n^{1/2})\, 2^{-\nu}, \qquad x \in Q_{\nu m}. \tag{3.8}$$

LEMMA 3.2. *Let* $d(x, F) \leqslant 2^{-(\nu-1)}$. *Then*

(i) $$|\tilde{f}_\nu^{(j)}(x)| \leqslant c2^\nu, \qquad |j| = 2,$$

(ii) $$|\tilde{f}_\nu^{(j)}(x)| \leqslant c\nu, \qquad |j| = 1,$$

(iii) $$|\tilde{f}_\nu(x)| \leqslant c.$$

Proof. Since $\tilde{f}_\nu(x) = \sum_i \varphi_{\nu i}(x)\, P_\nu(x, p_{\nu i})$, we see that $\tilde{f}_\nu^{(j)}(x)$ is a sum of terms of type

$$T_\nu^{l,m}(x) = \sum_i \varphi_{\nu i}^{(l)}(x)\, P_\nu^{(m)}(x, p_{\nu i}), \tag{3.9}$$

where l and m are multiindices with $l + m = j$.

Here, clearly, $P_\nu^{(m)}(x, p_{\nu_i}) = f_{m\nu}(p_{\nu i})$ if $|m| = 1$, and $P_\nu^{(m)}(x, p_{\nu i}) = 0$ if $|m| > 1$.

Now, let $b \in F$ be a point with $|x - b| = d(x, F)$.
Since $\sum_i \varphi_{\nu i} = 1$, and thus $\sum_i \varphi_{\nu i}^{(l)} = 0$, $l \neq 0$, we have

$$T_\nu^{l,m}(x) = \sum_i \varphi_{\nu i}^{(l)}(P_\nu^{(m)}(x, p_{\nu i}) - P_\nu^{(m)}(x, b)), \qquad l \neq 0. \tag{3.10}$$

Suppose next that $\varphi_{\nu i}(x) \neq 0$. Then, by the construction of $\varphi_{\nu i}$, x belongs to a cube touching $Q_{\nu i}$, and thus, by (3.8)

$$|x - p_{\nu i}| \leqslant c2^{-\nu}, \qquad \varphi_{\nu i}(x) \neq 0 \tag{3.11}$$

and consequently

$$|p_{\nu i} - b| \leqslant |p_{\nu i} - x| + |x - b| \leqslant c2^{-\nu}, \qquad \varphi_{\nu i}(x) \neq 0. \tag{3.12}$$

Consequently, using (3.4), (3.6), and (3.7) we get from (3.10) that $|T_\nu^{l,m}(x)| \leqslant c2^\nu$, $l + m = j$, $|j| = 2$, $l \neq 0$, which gives part (i) of the lemma. Similarly we get $|T_\nu^{l,m}(x)| \leqslant c$, $l + m = j$, $|j| = 1$, $l \neq 0$. If $l + m = j$, $|j| = 1$, $l = 0$, we instead combine (1.8) and (3.9) and get $|T_\nu^{l,m}(x)| \leqslant c\nu$, so we get part (ii) of the lemma. Finally, since by (1.7) and (1.8)

$$|P_\nu(x, p_{\nu i})| \leqslant |f_\nu(p_{\nu i})| + \Big|\sum_{|j|=1} (x - p_{\nu i})^j f_{j\nu}(p_{\nu i})\Big| \leqslant c + c2^{-\nu}\nu \leqslant c,$$

we have $|\tilde{f}_\nu(x)| \leqslant c$.

LEMMA 3.3. *Let* $d(x, F) \leqslant 2^{-(\nu-1)}$ *and* $\nu \geqslant \mu$. *Then there holds*

(i) $|\tilde{f}_\nu(x) - \tilde{f}_\mu(x)| \leqslant c2^{-\mu}$,

(ii) $|f(x) - \tilde{f}_\mu(x)| \leqslant c2^{-\mu}$, $\quad x \in F$,

(iii) $|\tilde{f}_\nu^{(j)}(x) - \tilde{f}_\mu^{(j)}(x)| \leqslant c2^{\nu-\mu}$, $\quad |j| = 1$.

Proof. Since $\sum_i \varphi_{\nu i} = 1$, we have

$$\tilde{f}_\nu(x) - \tilde{f}_\mu(x) = \sum_i \varphi_{\nu i}(x)\, P_\nu(x, p_{\nu i}) - \sum_k \varphi_{\mu k}(x)\, P_\mu(x, p_{\mu k})$$

$$= \sum_i \sum_k \varphi_{\nu i}(x)\, \varphi_{\mu k}(x)(P_\nu(x, p_{\nu i}) - P_\mu(x, p_{\mu k})).$$

Now, by (3.11) and (i) in Definition 1.1

$$| P_\nu(x, p_{\nu i}) - P_\mu(x, p_{\nu i})|$$
$$\leqslant |f_\nu(p_{\nu i}) - f_\mu(p_{\nu i})|$$
$$+ \sum_{|j|=1} | x - p_{\nu i} ||f_{j\nu}(p_{\nu i}) - f_{j\mu}(p_{\nu i})| \leqslant c2^{-\mu} + c2^{-\nu}2^{\nu-\mu} = c2^{-\mu},$$

if $\varphi_{\nu i}(x) \neq 0$. Since, by (3.11),

$$| p_{\nu i} - p_{\mu k} | \leqslant | p_{\nu i} - x | + | x - p_{\mu k} | \leqslant c2^{-\nu} + c2^{-\mu} \leqslant c2^{-\mu},$$

if $\varphi_{\nu i}(x) \neq 0$ and $\varphi_{\mu k}(x) \neq 0$, we obtain from (3.6) then that

$$| P_\mu(x, p_{\nu i}) - P_\mu(x, p_{\mu k})| \leqslant c2^{-\mu}.$$

These estimates clearly give $| P_\nu(x, p_{\nu i}) - P_\mu(x, p_{\mu k})| \leqslant c2^{-\mu}$ if $\varphi_{\nu i}(x) \neq 0$ and $\varphi_{\mu k}(x) \neq 0$, and since $\sum_i \varphi_{\nu i} = \sum_k \varphi_{\mu k} = 1$ it follows from the expression for $\tilde{f}_\nu(x) - \tilde{f}_\mu(x)$ above that part (i) of the lemma holds.

If $x \in F$, then

$$| f(x) - \tilde{f}_\mu(x)| = \Big| \sum_k \varphi_{\mu k}(x)(f(x) - P_\mu(x, p_{\mu k}))\Big|$$

$$= \Big| \sum_k \varphi_{\mu k}(x)(f(x) - f_\mu(x) + f_\mu(x) - P_\mu(x, p_{\mu k}))\Big| ,$$

which by parts (i) and (ii) in Definition 1.1 and (3.11) is less than $c2^{-\mu}$.

Finally, for $|j| = 1$, we have $\tilde{f}_\nu^{(j)}(x) - \tilde{f}_\mu^{(j)}(x) = A + T_\nu^{j,0} - T_\mu^{j,0}$, where

$$A = \sum_i \varphi_{\nu i}(x)\, P_\nu^{(j)}(x, p_{\nu i}) - \sum_k \varphi_{\mu k}(x)\, P_\mu^{(j)}(x, p_{\mu k})$$

and T is given by (3.9). In the proof of Lemma 3.2 we saw that $|T_\mu^{j,0}| \leqslant c$ and $|T_\nu^{j,0}| \leqslant c$, and exactly as in the proof of part (i) of this lemma we get $|A| \leqslant c2^{\nu-\mu}$.

LEMMA 3.4. *We have*

(a) $|Ef(x) - \tilde{f}_\mu(x)| \leqslant c2^{-\mu}, \quad d(x, F) \leqslant 2^{-(\mu+1)}, \quad \mu \geqslant 1,$

(b) $|(Ef)^{(j)}(x)| \leqslant c(d(x, F))^{-1}, \quad x \in \mathscr{C}F, \quad |j| = 2,$

(c) $|Ef(x)| \leqslant c, \quad x \in R^n.$

Proof. If $x \in F$, then (a) is just statement (ii) of Lemma 3.3. If $x \notin F$, say $x \in \Delta_\tau$, where $\tau \geqslant \mu + 1$, we obtain since $\sum \varphi_\nu = 1$ (recall also the other properties of φ_ν)

$$|Ef(x) - \tilde{f}_\mu(x)| = \left| \sum_{\nu=1}^{\infty} \varphi_\nu(x) \tilde{f}_\nu(x) - \tilde{f}_\mu(x) \right|$$

$$\leqslant \sum_{\nu=\tau-1}^{\tau+1} \varphi_\nu(x) |\tilde{f}_\nu(x) - \tilde{f}_\mu(x)| \leqslant c2^{-\mu},$$

where the last inequality is a consequence of (i) in Lemma 3.3. Thus (a) is proved.

To prove (b), let $x \in \Delta_\tau$, $\tau \geqslant 2$. (If $\tau < 0$, then (b) is trivial since then $Ef(x) = 0$ in Δ_τ and the cases $\tau = 0$ and $\tau = 1$ are treated in a straightforward manner.) If $|j| = 2$, then $(Ef)^{(j)}$ is a sum of terms of type $\sum \varphi_\nu^{(l)}(x) \tilde{f}_\nu^{(j-l)}(x)$. These are estimated by means of (i) of Lemma 3.2 if $l = 0$ and after subtracting $\tilde{f}_{\tau-1}^{(j-l)}(x)$, by means of (3.2) and (i) and (iii) of Lemma 3.3 if $l > 0$. One immediately obtains $|(Ef)^{(j)}(x)| < c2^\tau$, $x \in \Delta_\tau$, which proves (b).

Finally, (c) is a consequence of (iii) of Lemma 3.2.

3.5. It is now easy to prove that $Ef \in \Lambda(R^n)$. We shall prove that $|Ef| \leqslant c$, which is just statement (c) of Lemma 3.4, and that

$$|\Delta_h^2(Ef)(x)| \leqslant c\,|h|. \tag{3.13}$$

It is enough to prove (3.13) for, say, $|h| < 1/16$, since if $|h| > 1/16$, then (3.13) is a consequence of $|Ef| \leqslant c$. In order to prove (3.13) we consider two cases.

Case 1. $d(x, F) \geqslant 2\,|h|$. Then Ef is infinitely differentiable in a neighborhood of the line segment L between $x - h$ and $x + h$, and we obtain from the mean-value theorem and (b) of Lemma 3.4 that

$$|\Delta_h^2(Ef)(x)| \leqslant c\,|h|^2 \{d(L, F)\}^{-1} \leqslant c\,|h|.$$

Case 2. $d(x, F) < 2 \mid h \mid$. Choose μ so that

$$2^{-\mu-2} < d(x, F) + \mid h \mid \leqslant 2^{-\mu-1}.$$

From part (a) of Lemma 3.4 we obtain

$$\mid \Delta_h^2(Ef)(x) - \Delta_h^2\tilde{f}_\mu(x)\mid \leqslant c2^{-\mu} \leqslant c \mid h \mid,$$

where the last inequality is a consequence of $2^{-\mu-2} \leqslant 3 \mid h \mid$. From the mean-value theorem and part (i) of Lemma 3.2 we get

$$\mid \Delta_h^2\tilde{f}_\mu(x)\mid \leqslant c \mid h \mid^2 2^{\mu} \leqslant c \mid h \mid.$$

This proves (3.13).

In order to prove that Ef is continuous, let $x \in F$ and take $y \in R^n$ with $\mid x - y \mid \leqslant 2^{-(\mu+1)}$. Then, by (a) of Lemma 3.4, the mean-value theorem, and (ii) of Lemma 3.2,

$$\mid Ef(x) - Ef(y)\mid \leqslant c2^{-\mu} + \mid \tilde{f}_\mu(x) - \tilde{f}_\mu(y)\mid \leqslant c2^{-\mu} + c\mu 2^{-\mu}.$$

Thus, Ef is continuous at x, and since it is obvious from the definition of Ef that Ef is continuous outside F, we get that Ef is continuous in R^n.

4. Applications

The aim of this section is to consider some other possible definitions of $\Lambda_1(F)$ and to investigate whether they coincide with our Definition 1.1. In case of coincidence we get by means of Theorem 3.1 other characterizations of the restriction to F of the class $\Lambda(R^n)$. In this way we get (Corollary 4.2) a new proof of a well-known result.

4.1. We shall consider a set $F \subset R^n$ with boundary given by $x^n = \psi(x^1,..., x^{n-1})$, where $x = (x^1,..., x^n)$ and $\psi \in \mathrm{Lip}_1(M)$, i.e.,

$$\mid \psi(t) - \psi(t')\mid \leqslant M \mid t - t' \mid \qquad \text{for} \quad t, t' \in R^{n-1}.$$

Proposition 4.1. *Let $\psi \in \mathrm{Lip}_1(M)$ and let F be given by $F = \{x \in R^n$: $x^n \geqslant \psi(x^1,..., x^{n-1})\}$. Then $f \in \Lambda_1(F)$ if f is continuous on F and, for some constant M_1,*

$$\mid f(x)\mid \leqslant M_1, \qquad x \in F, \tag{4.1}$$

$$\mid \Delta_h^2 f(x)\mid \leqslant M_1 \mid h \mid, \tag{4.2}$$

when the line segment between $x - h$ and $x + h$ lies entirely in F. Moreover, the norm of f in $\Lambda_1(F)$ is less than cM_1 where c depends only on the dimension n

and the Lipschitz constant M. Conversely, if $f \in \Lambda_1(F)$, *then* (4.1) *and* (4.2) *are satisfied with* $M_1 \leqslant c \|f\|_{\Lambda_1(F)}$.

Remark 4.1. From the proof it will follow that in order to prove that $f \in \Lambda_1(F)$ it is enough to assume that (4.1) and (4.2) are true with F changed to the interior of F.

Remark 4.2. The alternative characterization of $\Lambda_1(F)$ given in Proposition 4.1 is, of course, simpler and more satisfactory than our original definition of $\Lambda_1(F)$. For general closed sets F it is, however, not true that $\Lambda_1(F)$ consists of all continuous bounded functions on F satisfying

$$|\Delta_h^2 f(x)| \leqslant M_1 |h| \quad \text{if} \quad x, x+h, x-h \in F. \tag{4.2'}$$

In fact, take any set F such that the three points x, $x - h$, and $x + h$, where $h \neq 0$, never belong to F simultaneously. The subset of R^1 consisting of zero and the points 3^{-k}, $k = 1, 2, \ldots$, is such a set. Then (4.2′) is automatically satisfied and $\Lambda_1(F)$ does not contain all bounded continuous functions on F since every $f \in \Lambda_1(F)$ can be extended to a function $Ef \in \Lambda(R^n)$ which satisfies $|Ef(x) - Ef(y)| \leqslant c\,|x - y| \cdot |\ln|x - y||$ (see the example in Remark 2.1).

In the proof we shall use the following lemma which is a special case of Whitney's extension theorem (see [10] or [8, Chap. VI, Theorem 4]).

LEMMA 4.1. *Let* $\{g^{(j)}\}_{|j|\leqslant 1}$, *with* $g^{(0)} = g$, *be defined on a closed set* $G \subset R^n$ *so that, for some constant* M_0 *and for all* $x, y \in G$, $|j| \leqslant 1$,

$$\left| g^{(j)}(x) - \sum_{|j+l|\leqslant 1} (x-y)^l g^{(j+l)}(y) \right| \leqslant M_0 |x-y|^{2-|j|}, \tag{4.3}$$

$$|g^{(j)}(x)| \leqslant M_0. \tag{4.4}$$

Then $g^{(0)} = g$ *can be extended to a function* $g \in C^1(R^n)$ *with the given functions* $g^{(j)}$, $|j| = 1$, *as the partial derivatives of g on G, such that* (4.3) *and* (4.4) *are true for all* $x, y \in R^n$ (*with* $g^{(j)}$, $|j| = 1$, *denoting the partial derivatives of g*) *if* M_0 *is replaced by* cM_0, *where c depends only on the dimension n.*

Proof of Proposition 4.1. The converse part follows, for instance, by means of Theorem 3.1. The proof of the direct part proceeds in several steps:

Step 1. Choose $F_\nu \subset F$, $\nu = 1, 2, \ldots$, so that F_ν is equal to F translated a distance $2^{-\nu}$ in the positive direction of the x^n-axis, i.e., the boundary ∂F_ν is given by the equation $x^n = \psi(x^1, \ldots, x^{n-1}) + 2^{-\nu}$. Then it follows easily (draw a figure; compare [8, Lemma 2, p. 182]) from the Lipschitz condition that, for some constant $M' > 0$ depending only on the Lipschitz constant M,

$$x \in \partial F_\nu \Rightarrow d(x, \partial F) \geqslant M' 2^{-\nu}, \tag{4.5}$$

where $d(x, \partial F)$ denotes the distance from x to ∂F. Furthermore, $\partial F_\nu \in \mathrm{Lip}_1(M)$, because ∂F_ν and ∂F satisfy the same Lipschitz condition since ∂F_ν is a translation of ∂F.

Step 2. Choose φ as in the proof of Proposition 2.1 but with $\varphi(x) = 0$ for $|x| \geqslant M'/2$ where M' is the constant in (4.5). Put $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$ and

$$f_\nu(x) = (f * \varphi_\nu)(x), \qquad x \in F_\nu . \tag{4.6}$$

Exactly as in the proof of Proposition 2.1 (see formula (2.2)) it is proved that $|D^j f_\nu(x)| \leqslant c 2^\nu$, $x \in F_\nu$, $|j| = 2$, and hence (compare, for $j = 0$ and $|j| = 1$, the first inequality in (2.4) and in (2.5), respectively)

$$\Big| f_\nu^{(j)}(x) - \sum_{|j+l| \leqslant 1} (x-y)^l f_\nu^{(j+l)}(y) \Big| \leqslant c 2^\nu |x - y|^{2-|j|},$$
$$x, y \in F_\nu, \ |j| \leqslant 1, \tag{4.7}$$

if all points of the line segment between x and y belong to F_ν. If some points of this line segment lie in the complement of F_ν, (4.7) is still true which is realized in the following way:

Take two points x' and y' in F_ν such that (1) the line segment between x and x' is parallel to the x^n-axis, the line between x' and y' to the R^{n-1}-plane, and the line between y' and y to the x^n-axis, (2) the polygon joining x, x', y', and y belongs to F_ν. The Lipschitz condition means that x' and y' may be chosen so that $|x - x'| \leqslant c|x - y|$, $|x' - y'| \leqslant c|x - y|$, and $|y' - y| \leqslant c|x - y|$. This gives, by means of the case when (4.7) is already proved,

$$\begin{aligned} |Df_\nu(x) - Df_\nu(y)| &\leqslant |Df_\nu(x) - Df_\nu(x')| + |Df_\nu(x') - Df_\nu(y')| \\ &\quad + |Df_\nu(y') - Df_\nu(y)| \leqslant c 2^\nu |x - y|. \end{aligned}$$

Analogously, by the cases already proved,

$$\begin{aligned} &|f_\nu(x) - f_\nu(y) - (x - y)\, Df_\nu(y)| \\ &\quad\leqslant |f_\nu(x) - f_\nu(x') - (x - x')\, Df_\nu(x')| \\ &\qquad + |-f_\nu(y') + f_\nu(x') + (y' - x')\, Df_\nu(x')| \\ &\qquad + |f_\nu(y') - f_\nu(y) - (y' - y)\, Df_\nu(y)| \\ &\qquad + |(x - y')(Df_\nu(x') - Df_\nu(y))| \leqslant c 2^\nu |x - y|^2, \end{aligned}$$

i.e., (4.7) is true for all $x, y \in F_\nu$.

We also note that (as in the proof of Proposition 2.1)

$$|f_\nu(x)| \leqslant c \quad \text{and} \quad |D^j f_\nu(x)| \leqslant c2^\nu, \qquad |j| = 1, \quad x \in F_\nu. \tag{4.8}$$

Step 3. So far, the functions f_ν are defined (by means of (4.6)) on F_ν only. We now use Lemma 4.1 for each fixed ν (with $G = F_\nu$, $g^{(j)} = f_\nu^{(j)}$, and $M_0 = c2^\nu$) and obtain extensions of f_ν from F_ν to functions $f_\nu \in C^1(R^n)$ so that

$$\text{(4.7) and (4.8) hold for all } x, y \in R^n. \tag{4.9}$$

We shall prove that $f \in \Lambda_1(F)$ by showing that the functions $f_{j\nu} = D^j f_\nu$, $|j| \leqslant 1$, $\nu = 1, 2, \ldots$, satisfy conditions (i)–(iii) in Definition 1.1. By (4.9), (ii) and (iii) are already verified. We shall verify (i) in Steps 4 and 5.

Step 4. Let $y = x - h \in F$, where h, $|h| \leqslant c2^{-\nu}$, is a point on the positive x^n-axis such that x and $x + h \in F_\nu$. Then

$$\begin{aligned} |f_\nu(y) - f(y)| &\leqslant |\Delta_h^2 f_\nu(x)| + |2f_\nu(x) - 2f(x)| \\ &\qquad + |f(x+h) - f_\nu(x+h)| + |\Delta_h^2 f(x)| \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

By the mean-value theorem and (4.9), $\mathrm{I} \leqslant c2^{-\nu}$. In the same way as in the proof of Proposition 2.1 it follows, because of (4.5), that $\mathrm{II} + \mathrm{III} \leqslant c2^{-\nu}$. Finally, $\mathrm{IV} \leqslant c2^{-\nu}$ by the assumption on f, and (1.1) is proved.

We also need the estimate

$$|f_\nu(y) - f_\mu(y)| \leqslant c2^{-\mu} \qquad \text{if} \quad d(y, F) \leqslant c2^{-\nu}, \quad \nu \geqslant \mu, \tag{4.10}$$

which is proved in the same way by replacing f by f_μ and estimating IV in the same way as I.

Step 5. The inequality

$$|D^j f_\nu(x) - D^j f_\mu(x)| \leqslant c2^{\nu-\mu}, \qquad \text{if} \quad x \in F, \quad \nu > \mu, \quad |j| = 1,$$

is proved, using (4.10), exactly like (2.6). Thus Proposition 4.1 is proved.

COROLLARY 4.1. *Let* $F = D \cup \partial D$ *where* D *is an open set in* R^n *with boundary* ∂D *which is minimally smooth in the sense used by Stein in* [8, Chap. VI, Sect. 3.3]. *Let* f *be continuous on* F *and satisfy conditions* (4.1) *and* (4.2) *in Proposition* 4.1. *Then* $f \in \Lambda_1(F)$ *with norm less than a constant (depending on* F*) times* M_1.

The proof proceeds by means of Proposition 4.1 and the method used in [8, Chap. VI, Sect. 3.3.1]. By combining Proposition 4.1 or Corollary 4.1 with Theorem 3.1 we obtain the following classical extension theorem [7, pp. 380–383] for the class $\Lambda(R^n)$.

COROLLARY 4.2. *Let F be as in Proposition* 4.1 *or Corollary* 4.1. *Let f be continuous on F and satisfy* (4.1) *and* (4.2) *in Propositoin* 4.1. *Then f can be extended from F to a function in $\Lambda(R^n)$ with norm less than a constant (depending on F) times M_1.*

Remark 4.3. Let D be a domain of the same type as in Corollary 4.1 and $\Lambda(D)$ the space of all continuous functions f satisfying (4.1) and (4.2) with F replaced by D. A characterization of $\Lambda(D)$ similar to our definition of $\Lambda_1(F)$ may be obtained from the theory of interpolation of linear operators. In [3] it is shown that for Lipschitz-graph domains D, the second-order modulus of smoothness $\omega_2(\delta, f)$ is equivalent to the K-functional

$$K_2(\delta, f) = \inf\{\|f - g\|_\infty + \delta \mid g \mid_{\infty,2} : g \in C^2(D)\},$$

where $\mid g \mid_{\infty,2} = \sup_{|j|=2} \| D^j{}_g \|_\infty$ and $\| \cdot \|_\infty$ denotes sup-norm in D. There are constants c_1, $c_2 > 0$ such that for $0 < \delta < 1$

$$c_1\omega_2(\delta, f) \leqslant K_2(\delta^2, f) \leqslant c_2\omega_2(\delta, f).$$

This gives that a function f belongs to $\Lambda(D)$ if and only if for $0 < \delta < 1$ there exist $g_\delta \in C^2(D)$ such that for $x \in D$

$$\begin{aligned}
&\text{(i)} \quad |f(x) - g_\delta(x)| \leqslant M\delta, \\
&\text{(ii)} \quad |D^j g_\delta(x)| \leqslant M\delta^{-1}, \qquad |j| = 2, \\
&\text{(iii)} \quad |f(x)| \leqslant M.
\end{aligned}$$

In particular, if $D = R^n$, $f \in \Lambda(R^n)$, and we put $f_\nu = g_\delta$ where $\delta = 2^{-\nu}$ we directly obtain that f and f_ν satisfy (1.1), (1.3), (1.4), and (1.5) in Definition 1.1. Compare also with (2.1) and (2.2) in the proof of Proposition 2.1.

4.2. In the next proposition we treat generalizations of the condition considered in Proposition 2.3 and in (2.15) to a rather general closed set F. We treat the case $n = 2$ which is already entirely typical. We shall consider a set $F \subset R^2$ of the following kind. Suppose that there are constants c_1, c_2, and v, $0 < v < \pi/3$, so that, for every $x_0 \in F$ and every $\nu = 1, 2, \ldots$, there are points x_1, $x_2 \in F$ such that

$$c_1 2^{-\nu} \leqslant | x_i - x_0 | \leqslant c_2 2^{-\nu}, \qquad i = 1, 2, \tag{4.11}$$

and

$$x_0, x_1, x_2 \text{ are } v\text{-uniformly affine independent in the sense (2.14).} \tag{4.12}$$

(The last condition of course disappears in the R^1 case.)

EXAMPLES. The sets F in Proposition 4.1 and the usual Cantor sets in R^2 are of this type.

PROPOSITION 4.2. *Let $F \subset R^2$ be of the kind described above. Then $f \in \Lambda_1(F)$ if and only if, for all $x \in F$ and some constants M and c,*

$$|f(x)| \leqslant M, \tag{4.13}$$

$$|f(x) - P(x)| \leqslant M \max_{0 \leqslant i,k \leqslant 2} |x_i - x_k|, \tag{4.14}$$

for $\max_{0\leqslant i\leqslant 2} |x - x_i| \leqslant c \max_{0\leqslant i,k\leqslant 2} |x_i - x_k|$, *for all ν-uniformly affine independent points $x_0, x_1, x_2 \in F$, if P is the first degree polynomial (in two variables) interpolating to f at x_0, x_1, and x_2. The norm of f in $\Lambda_1(F)$ is equivalent to the infimum of the constant M in* (4.13) *and* (4.14).

For the proof we need

LEMMA 4.2. *If a first degree polynomial P in two variables satisfy $|P(x)| \leqslant M$ at three points x_0, x_1, and x_2 satisfying* (4.11) *and* (4.12), *then*

$$|P(x)| \leqslant c_1 M \qquad \text{if} \quad \max_{0\leqslant i\leqslant 2} |x - x_i| \leqslant c2^{-\nu}$$

(*here c_1 depends on c but not on ν*).

Proof. If $x = x_0 + \alpha(x_1 - x_0) + \beta(x_2 - x_0)$, then

$$P(x) = P(x_0) + \alpha(P(x_1) - P(x_0)) + \beta(P(x_2) - P(x_0)),$$

which gives the lemma.

Proof of Proposition 4.2. The "only if" part follows from Theorem 3.1 combined with Proposition 2.3 and (2.15). In order to prove the "if" part we subdivide, for each $\nu = 1, 2, \ldots$, R^2 into a mesh M_ν of squares $\{S_{i\nu}\}$ with sides of length $2^{-\nu}$ parallel to the axes. We do this so that each $x \in R^2$ belongs to exactly one square in M_ν and so that the squares in $M_{\nu+1}$ are obtained by bisecting each square in M_ν into four squares.

Choose a point $x_0 = x_{0i\nu} \in F \cap S_{i\nu}$, if $F \cap S_{i\nu}$ is nonempty, and, after that, points $x_1 = x_{1i\nu} \in F$ and $x_2 = x_{2i\nu} \in F$ satisfying the conditions stated for x_0, x_1, x_2 in (4.11) and (4.12). Let $P_{i\nu}$ be the first-degree polynomial interpolating to f at $x_k = x_{ki\nu}$, $k = 0, 1, 2$, and put

$$f_{j\nu}(x) = P_{i\nu}^{(j)}(x) \qquad \text{for } x \in F \cap S_{i\nu}, |j| \leqslant 1. \tag{4.15}$$

The partial derivatives $P_{i\nu}^{(j)}(x)$, $|j| = 1$, are constants and can be expressed in the following way. Let $x_j' = x_{ji\nu}' = x_{0i\nu} + 2^{-\nu} e_j$, where e_j is the unit vector in the x^j-direction. Then

$$P_{i\nu}^{(j)}(x) = \frac{P_{i\nu}(x_j') - P_{i\nu}(x_0)}{2^{-\nu}}, \qquad x \in R^n, |j| = 1. \tag{4.16}$$

We shall prove that $f \in \Lambda_1(F)$ by showing that $\{f_{j\nu}\}$, $|j| \leqslant 1$, $\nu = 1, 2, \ldots$, satisfy the conditions in Definition 1.1. (We put $f_{0\nu} = f_\nu$.) First we note that, by assumption (4.14), for $x \in F \cap S_{i\nu}$

$$|f(x) - f_\nu(x)| = |f(x) - P_{i\nu}(x)| \leqslant c2^{-\nu}.$$

This gives (1.1). In order to prove (1.3) we assume that $x \in S_{i\nu} \cap F$ and $y \in S_{k\nu} \cap F$. Then, by (4.15),

$$\left| f_\nu(x) - f_\nu(y) - \sum_{|j|=1} (x-y)^j f_{j\nu}(y) \right|$$
$$\leqslant | P_{i\nu}(x) - P_{k\nu}(x)| + | P_{k\nu}(x) - P_{k\nu}(y) - (x-y)\, DP_{k\nu}(y)| .$$

The second term on the right-hand side is zero since the polynomial $P_{k\nu}$ is of first degree. By inserting $f(x)$ and using (4.14) we see that the first term is less than $c2^{-\nu}$ if $| x - y | \leqslant 2^{-\nu}$. This proves (1.3). Next we want to prove (1.4). Again we assume that $x \in S_{i\nu} \cap F$, $y \in S_{k\nu} \cap F$, and that $| x - y | \leqslant 2^{-\nu}$.

Then, by (4.15) and (4.16) (with $x_0 = x_{0i\nu}$), for $| j | = 1$,

$$\begin{aligned} f_{j\nu}(x) - f_{j\nu}(y) &= P_{i\nu}^{(j)}(x) - P_{k\nu}^{(j)}(y) \\ &= 2^\nu(P_{i\nu}(x_j') - P_{i\nu}(x_0)) - P_{k\nu}^{(j)}(y) \\ &= 2^\nu(P_{i\nu}(x_j') - P_{k\nu}(x_j')) + 2^\nu(P_{k\nu}(x_0) - P_{i\nu}(x_0)), \quad (4.17) \end{aligned}$$

since $P_{k\nu}^{(j)}$, $| j | = 1$, is constant. By (4.14), $P_{i\nu}$ and $P_{k\nu}$ differ from f, and hence from each other, by less than $c2^{-\nu}$ at the points $x_{ki\nu}$, $k = 0, 1, 2$. Hence, by Lemma 4.2, the last member of (4.17) is less than c, proving (1.4). The proof of (1.2) is very similar. In fact, let $x \in F \cap S_{i\nu} \cap S_{k\mu}$, $\nu > \mu$. As in the proof of (1.4) we get for $| j | = 1$,

$$|f_{j\nu}(x) - f_{j\mu}(x)| = | 2^\nu(P_{i\nu}(x_j') - P_{k\mu}(x_j')) + 2^\nu(P_{k\mu}(x_0) - P_{i\nu}(x_0))| \leqslant c2^{\nu-\mu},$$

proving (1.2). It remains to prove (1.5) and (1.6). We note that $|f_\nu| \leqslant c$ since $| P_{i\nu} | \leqslant c$ at x_0, x_1, and x_2 and hence at $x \in S_{i\nu}$ by Lemma 4.2. This argument and (4.16) also proves (1.6). So Proposition 4.2 is proved.

Remark 4.4. A definition of a smooth class on an arbitrary closed set F similar to the one in (4.13) and (4.14) has been used in approximation problems in the theory of analytic functions (Dzjadyk [2, p. 71]). For sets $F \subset R^1$ this definition gives a result similar to Proposition 4.2 by the same method of proof as above.

Remark 4.5. It is also possible to give an equivalent definition of $\Lambda_1(F)$ for any closed set F in terms of local polynomial approximation. Denote by

$Q = Q(x_0, \delta)$ a cube with center x_0 and side length δ. Define $\Lambda^*(F)$ as the set of all functions f such that for each $Q = Q(x_0, \delta)$ there is a polynomial P_Q of degree $\leqslant 1$ such that

(i) $|f(x) - P_Q(x)| \leqslant M\delta$, $x \in Q \cap F$,

(ii) if $Q \cap F \neq \varnothing$ and $Q' \cap F \neq \varnothing$, then $|P_Q(x) - P_{Q'}(x)| \leqslant M \max(\delta, \delta')$, $x \in Q \cap Q'$,

(iii) if $Q \cap F \neq \varnothing$ and Q has side length $\frac{1}{2}$, then $|P_Q(x)| \leqslant M$, $x \in Q$.

Then one can show that $\Lambda^*(F) = \Lambda_1(F)$. The proof of this is similar to the proof of Proposition 4.2. It may be noted that the conditions imposed on f in Definition 1.1 are in a sense weaker than the conditions in the alternative definitions of $\Lambda_1(F)$ given in this remark and in Remark 1.3, since it is easy to verify directly, that if $f \in \Lambda^*(F)$ or f satisfies the requirements in Remark 1.3 then f belongs to $\Lambda_1(F)$.

5. A More General Form of the Extension Theorem

In this section we briefly discuss the generalization of some of the previous results to spaces $\Lambda_k(F)$, $k > 1$.

We first define the class $\Lambda_k(R^n)$, $k \geqslant 1$, k integer (cf. Definition 2.1 and, e.g., [8, p. 145]). A function f belongs to $\Lambda_k(R^n)$ if f is $k - 1$ times continuously differentiable and $|\Delta_h^2 f^{(j)}(x)| \leqslant M|h|$, $|j| \leqslant k - 1$, $x, h \in R^n$, and $|f^{(j)}(x)| \leqslant M$, $|j| \leqslant k - 1$, $x \in R^n$. The norm of f is infimum of the constants M. Equivalently, f belongs to $\Lambda_k(R^n)$ if $|\Delta_h^{k+1} f(x)| \leqslant M|h|^k$ and $|f(x)| \leqslant M$, (see [7, p. 159]). Here $\Delta_h^{k+1} f(x)$ denotes the difference of order $k + 1$ with step h, i.e., $\Delta_h^1 f(x) = f(x + h) - f(x)$ and, for $k > 0$, $\Delta_h^{k+1} f(x) = \Delta_h^1(\Delta_h^k f)(x)$.

The following definition of $\Lambda_k(F)$ is a generalization of Definition 1.1. An element of $\Lambda_k(F)$ is a collection $\{f_j\}_{|j| \leqslant k-1}$, where j is a multiindex and the functions f_j may conveniently be considered as derivatives of f_0. We sometimes write f for this collection. Compare also the definition of $\mathrm{Lip}(\alpha, F)$, $\alpha > 0$, given in [8, p. 176].

Definition 5.1. Let F be a closed subset of R^n, let $a > 0$, let k be an integer, $k \geqslant 1$, and let the functions f_j, $|j| \leqslant k - 1$ be defined on F. Then we say that $\{f_j\}_{|j| \leqslant k-1}$ belongs to $\Lambda_k(F)$ if there exist collections $\{f_{j\nu}\}_{|j| \leqslant k}$, $\nu = 1, 2, 3, \ldots$, of functions defined on F such that for $x, y \in F$

$$\text{(i)} \qquad |f_j(x) - f_{j\nu}(x)| \leqslant M 2^{-\nu(k-|j|)}, \qquad \nu \geqslant 1, |j| \leqslant k - 1,$$

$$|f_{j\nu}(x) - f_{j\mu}(x)| \leqslant M 2^{\mu-\nu}, \qquad \mu \geqslant \nu \geqslant 1, |j| = k,$$

(ii) $$\left| f_{j\nu}(x) - \sum_{|j+l|\leqslant k} \frac{f_{j+l,\nu}(y)}{l!}(x-y)^l \right| \leqslant M2^{-\nu(k-|j|)},$$
$$|x-y| \leqslant a2^{-\nu},\ \nu \geqslant 1,\ |j| \leqslant k,$$

(iii) $$|f_{j1}(x)| \leqslant M, \qquad |j| \leqslant k.$$

As the norm $\|f\|_{\Lambda_k(F)}$ of f, we again take the infimum of all M such that the conditions (i)–(iii) are satisfied for some $\{f_{j\nu}\}_{|j|\leqslant k}$.

If $F = R^n$ and $\{f_j\}_{|j|\leqslant k-1} \in \Lambda_k(F)$, then it is readily verified that f_0 is $k-1$ times differentiable, and that the functions f_j are uniquely determined by f_0 by means of $f_j = f_0^{(j)}$. Furthermore, the analog of Proposition 2.1 holds, i.e., the space $\Lambda_k(F)$ as defined by Definition 5.1 is for $F = R^n$ equivalent to the classical space $\Lambda_k(R^n)$, defined before Definition 5.1. The proof is similar to the proof of Proposition 2.1, but a significant difference is that the function f_r in that proof shall be defined by

$$f_r(x) = \int \{(-1)^k \Delta_{t/(k+1)}^{k+1} f(x) + f(x)\}\, \varphi_r(t)\, dt.$$

The following theorem is our most general version of the extension theorem.

THEOREM 5.1. *Let F be a closed subset of R^n. Then every $f = \{f_j\}_{|j|\leqslant k-1} \in \Lambda_k(F)$ may be extended to a function Ef in $\Lambda_k(R^n)$. Ef is an extension of f in the sense that the restriction to F of the partial derivative $(Ef)^{(j)}$ is f_j for $|j| \leqslant k-1$. Furthermore, the extension can be made so that $\|Ef\|_{\Lambda_k(R^n)} \leqslant c\|f\|_{\Lambda_k(F)}$, where the constant c only depends on n and k, and so that Ef is infinitely differentiable outside F.*

To prove this theorem, we use the following more general form of the extension given in Section 3.3. Put

$$P_\nu(x, y) = \sum_{|j|\leqslant k} \frac{f_{j\nu}(y)}{j!}(x-y)^j$$

and let $\varphi_{\nu i}$ and $p_{\nu i}$ be as in Section 3. Then we define Ef by

$$Ef(x) = \sum_{\nu=1}^{\infty} \varphi_\nu(x) \tilde{f}_\nu(x),\ x \in \mathscr{C}F, \qquad \text{and} \qquad Ef(x) = f_0(x),\ x \in F,$$

where $\tilde{f}_\nu$ is given by

$$\tilde{f}_\nu(x) = \sum_i \varphi_{\nu i}(x)\, P_\nu(x, p_{\nu i}).$$

We omit the proof of the fact that Ef belongs to $\Lambda_k(R^n)$, since the proof of this in the case $k = 1$ already given in Section 3 is almost entirely typical for the general case.

Note however that the expression for $P_\nu(x, q) - P_\nu(x, r)$ given in Section 3.4 shall be replaced by a more general lemma given, e.g., in [8, p. 177].

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